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Abstract

We extend the definition of generalized parity P, charge-conjugation C and time-reversal T operators to nondiagonalizable pseudo-Hermitian Hamiltonians, and use these generalized operators to describe the full set of symmetries of a pseudo-Hermitian Hamiltonian according to a fourfold classification. In particular, we show that TP and CTP are the generators of the P-antiunitary symmetries; moreover, a necessary and sufficient condition is provided for a pseudo-Hermitian Hamiltonian H to admit a P-reflecting symmetry which generates the P-pseudounitary and the P-pseudoantiunitary symmetries. Finally, a physical example is considered and some hints on the P-unitary evolution of a physical system are also given.

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1. Introduction

The studies on the *pseudo-Hermitian operators*, i.e., those operators which satisfy

$$\eta H \eta^{-1} = H^\dagger \tag{1}$$

with a non-unique $\eta = \eta^\dagger$, have recently developed along two seemingly uncorrelated lines.

On the one hand, starting from a stimulating paper by Bender *et al* [1] on the class of PT-symmetric Hamiltonians

$$H_\nu = P^2 + x^2 (\mathrm{i} x)^\nu \qquad \nu \geqslant 0$$

it has been proved, with a growing level of generalization [2–6] that one can associate with any pseudo-Hermitian Hamiltonian with a discrete spectrum a triple of operators P, C and T (also called, respectively, *generalized parity*, *charge-conjugation* and *generalized time-reversal* operators) with many interesting properties: among them, the possibility of obtaining (if any) a positive definite inner product.

On the other hand, some interest has been devoted to the study of the η-*unitary operators*, i.e., those operators U which satisfy

$$U^\dagger \eta U = \eta \qquad (2)$$

(with $\eta = \eta^\dagger$), and of their spectrum [7, 8]. This concept arises in a very natural way in connection with the pseudo-Hermiticity property, in the sense that for any η-pseudo-Hermitian Hamiltonian H, $U = e^{iH}$ is trivially η-unitary.

In this context, we intend to extend the definition of generalized parity, charge-conjugation and generalized time-reversal operators to the class of nondiagonalizable pseudo-Hermitian Hamiltonians, which can sometimes occur in physics (for instance, they can be obtained from the diagonalizable ones for some critical parameter values).

At the same time, these generalized operators will also be used in order to describe the *full* set of symmetries of a pseudo-Hermitian Hamiltonian (according to a fourfold classification), proving that a very deep connection exists between the two topics above.

In order to achieve this twofold goal, we premise in section 2 a proposition which provides a necessary and sufficient condition for a non-Hermitian operator with discrete spectrum to admit a linear, involutory symmetry. Next, we define in section 3 two families of generalized parity and charge-conjugation operators $\{P_\sigma\}$ and $\{C_\sigma\}$, respectively, and a (antilinear) generalized time-reversal operator T associated with a nondiagonalizable pseudo-Hermitian Hamiltonian H, showing that H is P_σ-pseudo-Hermitian, and that C_σ, TP_σ and $C_\sigma T P_{\sigma'}$ are involutory symmetries of H.

In section 4 we begin the study of the indefinite inner product spaces (actually, Krein spaces) that can be obtained by considering a new (possibly indefinite) inner product in our Hilbert space. In section 5 we first recall a previous, exhaustive classification of the symmetries S of Krein spaces [9], which brings into consideration the P-pseudounitary, P-antiunitary and P-pseudoantiunitary operators, besides the P-unitary ones. In particular, we consider the subset S_H of the elements of S which commute with H, and show that the P-antiunitary symmetries, both in S and in S_H, can be generated by some of the generalized operators previously introduced, namely TP and CTP. Moreover, a necessary and sufficient condition is provided for a (possibly) nondiagonalizable pseudo-Hermitian Hamiltonian H to admit a P-reflecting symmetry which generates the P-pseudounitary and P-pseudoantiunitary symmetries in S_H, and a possible physical meaning of such an operator is suggested.

Finally, in section 6 a physical model is considered which allows us to illustrate all the above results (and some hints on the P-unitary evolution of a physical system are also given), whereas section 7 contains some concluding remarks.

2. Non-Hermitian operators and linear involutory symmetries

Following [10, 11], we consider here only linear operators H acting in a separable Hilbert space $\mathfrak{H}$ and having a discrete spectrum. Moreover, throughout this paper we shall assume that all the eigenvalues E_n of H have finite algebraic multiplicity g_n and that there is a basis of $\mathfrak{H}$ in which H is block diagonal with finite-dimensional diagonal blocks. Then, a complete biorthonormal basis $\mathfrak{E} = \{|\psi_n, a, i\rangle, |\phi_n, a, i\rangle\}$ exists such that the operator H can be written in the following form [10]:

$$H = \sum_n \sum_{a=1}^{d_n} \left(E_n \sum_{i=1}^{p_{n,a}} |\psi_n, a, i\rangle\langle\phi_n, a, i| + \sum_{i=1}^{p_{n,a}-1} |\psi_n, a, i\rangle\langle\phi_n, a, i+1| \right) \qquad (3)$$

where d_n denotes the geometric multiplicity (i.e., the degree of degeneracy) of E_n, a is a degeneracy label, $p_{n,a}$ represents the dimension of the simple Jordan block $J_a(E_n)$ associated with the labels n and a (hence, $\sum_{a=1}^{d_n} p_{n,a} = g_n$).

$|\psi_n, a, 1\rangle$ (respectively, $|\phi_n, a, p_{n,a}\rangle$) is an eigenvector of H (respectively, $H^\dagger$):

$$H|\psi_n, a, 1\rangle = E_n|\psi_n, a, 1\rangle \qquad H^\dagger|\phi_n, a, p_{n,a}\rangle = E_n^*|\phi_n, a, p_{n,a}\rangle \tag{4}$$

and the following relations hold:

$$H|\psi_n, a, i\rangle = E_n|\psi_n, a, i\rangle + |\psi_n, a, i-1\rangle \qquad i \neq 1 \tag{5}$$

$$H^\dagger|\phi_n, a, i\rangle = E_n^*|\phi_n, a, i\rangle + |\phi_n, a, i+1\rangle \qquad i \neq p_{n,a}. \tag{6}$$

The elements of the biorthonormal basis obey the usual relations

$$\langle \psi_m, a, i|\phi_n, b, j\rangle = \delta_{mn}\delta_{ab}\delta_{ij} \tag{7}$$

$$\sum_n \sum_{a=1}^{d_n} \sum_{i=1}^{p_{n,a}} |\psi_n, a, i\rangle\langle\phi_n, a, i| = \sum_n \sum_{a=1}^{d_n} \sum_{i=1}^{p_{n,a}} |\phi_n, a, i\rangle\langle\psi_n, a, i| = \mathbf{1}. \tag{8}$$

Let us now prove a necessary and sufficient condition for a non-Hermitian Hamiltonian to admit a linear involutory symmetry (we recall that a linear operator C is called *involutory* whenever $C^2 = \mathbf{1}$).

Proposition 1. *Let H be a linear operator with discrete spectrum. Then an involutory (non-trivial) linear operator C exists such that $[H, C] = 0$, if and only if H admits at least two linearly independent eigenvectors.*

Proof. Let us suppose that a non-trivial involutory operator C exists such that $[H, C] = 0$ and that H admits only one eigenvector, namely $|1\rangle$

$$H|1\rangle = E|1\rangle. \tag{9}$$

Then, a basis $\{|i\rangle\}$ exists in which

$$H|i\rangle = E|i\rangle + |i-1\rangle \qquad i \neq 1. \tag{10}$$

Multiplying the left-hand side of equation (9) by C, and recalling that $[H, C] = 0$ and $C^2 = \mathbf{1}$, one easily obtains

$$C|1\rangle = \epsilon|1\rangle$$

where $\epsilon = \pm 1$. In the same manner, from equation (10) (with $i = 2$), it follows that

$$HC|2\rangle = EC|2\rangle + \epsilon|1\rangle \tag{11}$$

and, linearly combining equations (11) and (10) (with $i = 2$),

$$H(\epsilon C - \mathbf{1})|2\rangle = E(\epsilon C - \mathbf{1})|2\rangle$$

i.e., $(\epsilon C - \mathbf{1})|2\rangle$ is an eigenvector of H. Hence,

$$(\epsilon C - \mathbf{1})|2\rangle = \beta|1\rangle \qquad (\beta \in \mathbb{C}).$$

On the other hand,

$$0 = (\epsilon C + \mathbf{1})(\epsilon C - \mathbf{1})|2\rangle = (\epsilon C + \mathbf{1})\beta|1\rangle = 2\beta|1\rangle$$

which implies $C|2\rangle = \epsilon|2\rangle$. Iterating the above procedure again and again, we conclude that $C = \epsilon\mathbf{1}$.

In order to prove the converse implication, let equation (3) hold, with $\sum_n d_n \geqslant 2$; then the following involutory (non-trivial) operator:

$$C_\sigma = \sum_n \sum_{a=1}^{d_n} \sum_{i=1}^{p_{n,a}} \sigma_n^a |\psi_n, a, i\rangle \langle \phi_n, a, i| \tag{12}$$

(where $\sigma = \{\sigma_n^a\}$ denotes an arbitrary sequence of signs $\sigma_n^a = \pm$ which depend only on the n and a indices) commutes with H. $\square$

Finally, the following useful property of involutory operators holds.

Proposition 2. *Every linear, involutory operator C is similar to a Hermitian operator.*

Proof. Let M be a linear (in general, non-unitary) invertible transformation which reduces C to its Jordan canonical form J

$$M^{-1}CM = J = D + N$$

where D is a (real, involutory) diagonal matrix, N is an upper triangular matrix and $[D, N] = 0$.
    Then,

$$J^2 = D^2 + 2ND + N^2 = \mathbf{1}$$

hence

$$N(2D + N) = \mathbf{0}.$$

Now, observe that $2D + N$ is invertible, since $\det(2D + N) = \det 2D \neq 0$; then, $N = \mathbf{0}$ and the thesis follows at once. $\square$

3. The generalized C, TP and CTP symmetries

In this section we will extend to the class of nondiagonalizable pseudo-Hermitian Hamiltonians the concepts of generalized parity P, charge-conjugation C and time-reversal T operators that have already proved to be very fruitful in the diagonalizable case.

At the moment we do not dwell upon the possible physical meaning of such operators, which, however, has been clearly discussed in the literature (see in particular [12] where these concepts have been successfully applied to describe Klein–Gordon fields). Our main goal is to obtain, for any pseudo-Hermitian Hamiltonian, a set of linear and antilinear symmetries, whose role will be enlightened in the following sections.

With reference to the same notation adopted above, let us use the subscript '0' to denote the real eigenvalues of H, and the subscripts '$\pm$' to denote the complex eigenvalues with positive or negative imaginary parts, respectively. Furthermore, we recall that for a pseudo-Hermitian operator the geometric multiplicity and the Jordan dimensions of the complex conjugate eigenvalues coincide (i.e., $d_{n_+} = d_{n_-}$ and $p_{n_+} = p_{n_-}$) [10]. Then, H assumes the following form:

$$H = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \left(E_{n_0} \sum_{i=1}^{p_{n_0,a}} |\psi_{n_0}, a, i\rangle\langle\phi_{n_0}, a, i| + \sum_{i=1}^{p_{n_0,a}-1} |\psi_{n_0}, a, i\rangle\langle\phi_{n_0}, a, i+1| \right)$$

$$+ \sum_{n_+,n_-} \sum_{a=1}^{d_{n_+}} \left[\sum_{i=1}^{p_{n_+,a}} \left(E_{n_+} |\psi_{n_+}, a, i\rangle\langle\phi_{n_+}, a, i| + E_{n_-} |\psi_{n_-}, a, i\rangle\langle\phi_{n_-}, a, i| \right) \right.$$

$$\left. + \sum_{i=1}^{p_{n_+,a}-1} \left(|\psi_{n_+}, a, i\rangle\langle\phi_{n_+}, a, i+1| + |\psi_{n_-}, a, i\rangle\langle\phi_{n_-}, a, i+1| \right) \right]. \tag{13}$$

We can associate with H a family $\{P_\sigma\}$ of operators defined as follows:

$$P_\sigma := \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sum_{i=1}^{p_{n_0,a}} \sigma_{n_0}^a |\phi_{n_0}, a, p_{n_0,a} + 1 - i\rangle\langle\phi_{n_0}, a, i|$$

$$+ \sum_{n_+, n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} \sigma_{n_+}^a \big(|\phi_{n_+}, a, p_{n_+,a} + 1 - i\rangle\langle\phi_{n_-}, a, i|$$

$$+ |\phi_{n_-}, a, p_{n_+,a} + 1 - i\rangle\langle\phi_{n_+}, a, i|\big) \tag{14}$$

(where $\sigma = \{\sigma_n^a\}$ denotes an arbitrary sequence of signs $\sigma_n^a = \pm$ which depend only on the n and a indices).

It is easy to verify that any P_σ is a Hermitian operator; furthermore H is a P_σ-pseudo-Hermitian operator, that is it satisfies the relation

$$P_\sigma H P_\sigma^{-1} = H^\dagger.$$

Observe that the generalized parity P_σ operator given in equation (14) generalizes from various points of view the generalized parity operators introduced in [2, 4, 5]. Indeed, as we said above, it is defined for a (possibly) nondiagonalizable pseudo-Hermitian operator H, and it reduces to that introduced in [5] and to the inverse of [4] whenever a diagonalizable Hamiltonian is taken into account.

Moreover, if the Hamiltonian H is Hermitian with non-degenerate spectrum, then the generalized parity operator P_σ given in equation (14) satisfies the following properties: (i) P_σ is linear and Hermitian; (ii) P_σ commutes with H; (iii) $P_\sigma^2 = \mathbf{1}$; (iv) the nth eigenstate of H is also an eigenstate of P_σ with eigenvalue $(-1)^n$. Hence our generalized parity operator has the same formal properties as the generalized parity operator introduced in [2].

Analogously, we can associate with H a family of generalized charge-conjugation operators given by

$$C_\sigma := \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sum_{i=1}^{p_{n_0,a}} \sigma_{n_0}^a |\psi_{n_0}, a, i\rangle\langle\phi_{n_0}, a, i|$$

$$+ \sum_{n_+, n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} \sigma_{n_+}^a \big(|\psi_{n_+}, a, i\rangle\langle\phi_{n_+}, a, i| + |\psi_{n_-}, a, i\rangle\langle\phi_{n_-}, a, i|\big) \tag{15}$$

and a time-reversal operator T

$$T := \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sum_{i=1}^{p_{n_0,a}} |\psi_{n_0}, a, i\rangle K\langle\psi_{n_0}, a, p_{n_0,a} + 1 - i| + \sum_{n_+, n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} \big(|\psi_{n_+}, a, i\rangle$$

$$\times K\langle\psi_{n_+}, a, p_{n_+,a} + 1 - i| + |\psi_{n_-}, a, i\rangle K\langle\psi_{n_-}, a, p_{n_-,a} + 1 - i|\big) \tag{16}$$

where K denotes the operation of the complex conjugation of numbers.

Comparing equations (12) and (15) one immediately sees that the latter is a particular case of the former, hence C_σ is an involutory symmetry of H.

The antilinear operator T satisfies the following remarkable conditions:

$$T = T^\dagger \qquad \text{and} \qquad T H^\dagger T^{-1} = H.$$

Moreover T generalizes the time-reversal operator introduced in [4]; indeed, whenever H is diagonalizable with a non-degenerate spectrum, the operator in equation (16) coincides with the inverse of the generalized time-reversal operator introduced in [4]. Besides, whenever H is Hermitian, the generalized time-reversal operator T defined in equation (16) satisfies the

following properties: (i) T is antiunitary; (ii) T commutes with H; (iii) $T^2 = \mathbf{1}$; (iv) the T symmetry is not spontaneously broken. Hence our generalized time-reversal operator has the same formal properties as the generalized time-reversal operator introduced in [3].

By using P_σ, T and C_σ, two (families of) involutory antilinear symmetries TP_σ and $C_\sigma TP_{\sigma'}$ of H can be constructed

$$
TP_\sigma = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sum_{i=1}^{p_{n_0,a}} \sigma_{n_0}^a |\psi_{n_0}, a, i\rangle K \langle \phi_{n_0}, a, i|
$$
$$
+ \sum_{n_+, n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} \sigma_{n_+}^a \left(|\psi_{n_+}, a, i\rangle K \langle \phi_{n_-}, a, i| + |\psi_{n_-}, a, i\rangle K \langle \phi_{n_+}, a, i| \right) \tag{17}
$$

and

$$
C_\sigma TP_{\sigma'} = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sum_{i=1}^{p_{n_0,a}} \sigma_{n_0}^{'a} \sigma_{n_0}^a |\psi_{n_0}, a, i\rangle K \langle \phi_{n_0}, a, i|
$$
$$
+ \sum_{n_+, n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} \sigma_{n_+}^{'a} \sigma_{n_+}^a \left(|\psi_{n_+}, a, i\rangle K \langle \phi_{n_-}, a, i| + |\psi_{n_-}, a, i\rangle K \langle \phi_{n+}, a, i| \right). \tag{18}
$$

By a direct inspection from equations (13), (15), (17) and (18) one easily verifies that

$$
[C_\sigma, TP_{\sigma'}] = 0 \tag{19}
$$

$$
[C_\sigma, H] = 0 \qquad (C_\sigma)^2 = \mathbf{1} \tag{20}
$$

$$
[TP_\sigma, H] = 0 \qquad (TP_\sigma)^2 = \mathbf{1} \tag{21}
$$

$$
[C_\sigma TP_{\sigma'}, H] = 0 \qquad (C_\sigma TP_{\sigma'})^2 = \mathbf{1}. \tag{22}
$$

Note that if H admits a real spectrum, then its TP_σ and $C_\sigma TP_{\sigma'}$ symmetries are not spontaneously broken in the following sense:

$$
TP_\sigma |\psi_{n_0}, a, i\rangle = \sigma_{n_0}^a |\psi_{n_0}, a, i\rangle \qquad C_\sigma TP_{\sigma'} |\psi_{n_0}, a, i\rangle = \sigma_{n_0}^{'a} \sigma_{n_0}^a |\psi_{n_0}, a, i\rangle.
$$

Finally, we recall that the following theorem holds [11].

Theorem 1. *Let H be an operator with a discrete spectrum. Then, there exists a definite operator η such that H is η-pseudo-Hermitian if and only if H is diagonalizable with a real spectrum.*

According to such a theorem, for any diagonalizable operator H with a real spectrum, such a positive definite operator can be built up by means of the vectors of the biorthonormal basis associated with H. Now, it is worthwhile observing that such an operator coincides with the product P_+ of a suitable couple of generalized parity and charge-conjugation operators:

$$
P_+ = P_\sigma C_\sigma = \sum_{n_0} \sum_{a=1}^{d_{n_0}} |\phi_{n_0}, a\rangle\langle \phi_{n_0}, a| \tag{23}
$$

(note that the same (arbitrary) sequence of signs must be chosen both in P_σ and in C_σ).

Let us conclude this section by some further remarks about the connection between the pseudo-Hermiticity property and the time-reversal symmetry. It was already known [13, 14] that for any diagonalizable Hamiltonian H an antilinear (involutory) operator exists which commutes with H if and only if H is pseudo-Hermitian. This result has been recently

generalized to the nondiagonalizable case [11], and the discussion above confirms it (see in particular equation (21)). This allows one to conclude that any time-reversal invariant (not necessarily diagonalizable) Hamiltonian must belong to the class of pseudo-Hermitian Hamiltonians.

The converse does not hold in general. Indeed, one can prove that an antilinear involutory symmetry is associated with any pseudo-Hermitian Hamiltonian, but in general one cannot interpret it as the '*physical*' time-reversal operator Θ; for instance, in the case of fermionic systems it is well known that

$$\Theta^2 = -\mathbf{1}$$

and the above results do not ensure the existence of such a symmetry. In order to go more deeply into the matter, we recall that the following theorem holds [11].

Theorem 2. *Let H be a linear operator with a discrete spectrum. Then, the following conditions are equivalent:*

(i) an antilinear operator $\mathfrak{T}$ exists such that $[H, \mathfrak{T}] = 0$, with $\mathfrak{T}^2 = -\mathbf{1}$;
(ii) H is pseudo-Hermitian and the Jordan blocks associated with any real eigenvalue occur in pairs.

From this theorem it follows in particular that whenever a pseudo-Hermitian operator H admits an antilinear symmetry $\mathfrak{T}$ with $\mathfrak{T}^2 = -\mathbf{1}$, both the geometric and the algebraic multiplicities of any real eigenvalue of H are even.

4. *P*-pseudo-Hermitian Hamiltonians and indefinite inner products

A very intriguing feature of the pseudo-Hermiticity property is the chance of slightly generalizing the usual quantum mechanical description of a physical system, by adopting a more general criterion for the representation of physical observables.

Indeed, as is well known, when one considers a η-pseudo-Hermitian operator H a new (possibly, but not necessarily) indefinite inner product can be defined in the Hilbert space $\mathcal{H}$ [15]:

$$\langle\!\langle \psi, \phi \rangle\!\rangle_\eta := \langle \psi | \eta | \phi \rangle \tag{24}$$

with respect to which H is self-adjoint, in the sense that $H^{\ddagger} := \eta^{-1} H^{\dagger} \eta = H$, hence $\langle\!\langle H\psi, \phi \rangle\!\rangle_\eta = \langle\!\langle \psi, H\phi \rangle\!\rangle_\eta$.

Let us then consider the orthogonal projectors $\Pi^{(\pm)}$ mapping $\mathcal{H}$ on the spans of the eigenvectors of η associated with its positive, or respectively negative, eigenvalues and observe that, η being invertible, none of its eigenvalues vanishes. Hence, trivially,

$$\Pi^{(+)} + \Pi^{(-)} = \mathbf{1}$$

and

$$\mathcal{H} = \mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)}$$

where $\mathcal{H}^{(\pm)} \equiv \Pi^{(\pm)}\mathcal{H}$. Moreover, it is easily seen that, for all vectors $|\chi^{\pm}\rangle \in \mathcal{H}^{(\pm)}$, the real number $\pm\langle\!\langle \chi^{\pm}, \chi^{\pm} \rangle\!\rangle_\eta$ is positive, and that $\mathcal{H}^{(\pm)}$ are Hilbert spaces relative to the norms $\|\chi^{\pm}\| = (\pm\langle\!\langle \chi^{\pm}, \chi^{\pm} \rangle\!\rangle_\eta)^{\frac{1}{2}}$, so that the space $\mathcal{H}$, endowed with the metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\eta$, is a *Krein space* [16, 17].

The Krein construction is particularly suitable in order to describe physical systems associated with pseudo-Hermitian Hamiltonians. Indeed, if the Hamiltonian operator H,

which determines the time dependence of the state vector $|\psi\rangle$ according to the Schrodinger equation

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

is η-pseudo-Hermitian, the conservation of the normalization of the state vectors with time still holds in a Krein space, where the new inner product is used [15].

More generally, it must be emphasized that *none of the requirements for a proper quantum mechanical interpretation is violated as long as pseudo-Hermitian Hamiltonians and, at the same time, a new positive definite inner product are employed* [18] (see also section 6 for some concrete examples).

Now, if we perform a linear (not necessarily unitary) transformation of the coordinate system in the Krein space $\mathcal{H}$ introduced above [19]:

$$|\widetilde{\psi}\rangle = S^{-1}|\psi\rangle$$

we have to put

$$\widetilde{\eta} = S^\dagger \eta S \tag{25}$$

in order to keep the length of the vector in $\mathcal{H}$ constant; $\tilde{\eta}$ is said to be *congruent* to η. (In particular, note that whenever η is a positive definite operator, for instance $\eta \equiv P_+$ (see equation (23)), a transformation exists such that $\tilde{P}_+ = \mathbf{1}$. Hence, if we restrict H to be diagonalizable with a real spectrum, we can conclude that *pseudo-Hermiticity is equivalent to Hermiticity* [20].)

The observables such as A and their adjoints are then transformed according to

$$\widetilde{A} = S^{-1}AS \qquad \widetilde{A}^{\ddagger} = \widetilde{\eta}^{-1}\widetilde{A}^\dagger\widetilde{\eta} = S^{-1}A^{\ddagger}S \tag{26}$$

in order to make their expectation values invariant [19]:

$$\langle A\rangle_\eta = \langle\psi|\eta A|\psi\rangle = \langle\widetilde{\psi}|\widetilde{\eta}\widetilde{A}|\widetilde{\psi}\rangle.$$

Let us consider now a P_σ-pseudo-Hermitian operator H, where P_σ is given in equation (14), and let us fix in $\mathcal{H}$ the (indefinite) inner product induced by P_σ.

Given any complete orthonormal basis $\mathfrak{F} = \{|u_n, a, i\rangle\}$ (that we denote by the same labels n, a, i used for the elements of $\mathfrak{E}$), let us suppose

$$S = \sum_n \sum_{a=1}^{d_n} \sum_{i=1}^{p_{n,a}} |\psi_n, a, i\rangle\langle u_n, a, i| \quad \left(\text{hence, } S^{-1} = \sum_n \sum_{a=1}^{d_n} \sum_{i=1}^{p_{n,a}} |u_n, a, i\rangle\langle \phi_n, a, i|\right). \tag{27}$$

Performing such a transformation one obtains, according to equation (26)

$$S^{-1}HS = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \left(E_{n_0} \sum_{i=1}^{p_{n_0,a}} |u_{n_0}, a, i\rangle\langle u_{n_0}, a, i| + \sum_{i=1}^{p_{n_0,a}-1} |u_{n_0}, a, i\rangle\langle u_{n_0}, a, i+1|\right)$$

$$+ \sum_{n_+,n_-} \sum_{a=1}^{d_{n_+}} \left[\sum_{i=1}^{p_{n_+,a}} \left(E_{n_+}^*|u_{n_+}, a, i\rangle\langle u_{n_+}, a, i| + E_{n_+}|u_{n_-}, a, i\rangle\langle u_{n_-}, a, i|\right)\right.$$

$$\left. + \sum_{i=1}^{p_{n_+,a}-1} \left(|u_{n_+}, a, i\rangle\langle u_{n_+}, a, i+1| + |u_{n_-}, a, i\rangle\langle u_{n_-}, a, i+1|\right)\right] = \tilde{H} \tag{28}$$

and, trivially,

$$\tilde{H}|u_n, a, 1\rangle = E_n|u_n, a, 1\rangle$$
$$\tilde{H}|u_n, a, i\rangle = E_n|u_n, a, i\rangle + |u_n, a, i-1\rangle \qquad i \neq 1. \tag{29}$$

Moreover, according to equation (25) the operator P_σ is transformed to the involutory, Hermitian operator

$$\widetilde{P_\sigma} = \widetilde{P_\sigma}^\dagger = S^\dagger P_\sigma S = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sum_{i=1}^{p_{n_0,a}} \sigma_{n_0}^a |u_{n_0}, a, p_{n_0,a} + 1 - i\rangle\langle u_{n_0}, a, i|$$

$$+ \sum_{n_+,n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} \sigma_{n_+}^a \left(|u_{n_+}, a, p_{n_+,a} + 1 - i\rangle\langle u_{n_-}, a, i| \right.$$

$$+ |u_{n_-}, a, p_{n_-,a} + 1 - i\rangle\langle u_{n_+}, a, i| \Big) \tag{30}$$

while both C_σ and $T P_\sigma$ are transformed according to (26), so that

$$\widetilde{C_\sigma} = \widetilde{C_\sigma}^\dagger = S^{-1} C_\sigma S = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sum_{i=1}^{p_{n_0,a}} \sigma_{n_0}^a |u_{n_0}, a, i\rangle\langle u_{n_0}, a, i|$$

$$+ \sum_{n_+,n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} \sigma_{n_+}^a \left(|u_{n_+}, a, i\rangle\langle u_{n_+}, a, i| + |u_{n_-}, a, i\rangle\langle u_{n_-}, a, i| \right) \tag{31}$$

and

$$\widetilde{T} = \widetilde{T}^\dagger = S^{-1} T S^{\dagger-1} = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sum_{i=1}^{p_{n_0,a}} |u_{n_0}, a, i\rangle K \langle u_{n_0}, a, p_{n_0,a} + 1 - i|$$

$$+ \sum_{n_+,n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} \left(|u_{n_+}, a, i\rangle K \langle u_{n_+}, a, p_{n_+,a} + 1 - i| \right.$$

$$+ |u_{n_-}, a, i\rangle K \langle u_{n_-}, a, p_{n_-,a} + 1 - i| \Big). \tag{32}$$

By inspection of equations (30), (31) and (32) one immediately realizes that $\tilde{T}$ and, obviously, $\widetilde{C_\sigma}$ are involutory: $\widetilde{C_\sigma}^2 = \tilde{T}^2 = \mathbf{1}$, and that $\tilde{T} \tilde{H}^\dagger \tilde{T} = \tilde{H}$. Moreover, all three of $\widetilde{C_\sigma}$, $\widetilde{P_\sigma}$ and $\tilde{T}$ mutually commute:

$$[\widetilde{P_\sigma}, \tilde{T}] = [\widetilde{P_\sigma}, \widetilde{C_{\sigma'}}] = [\widetilde{C_\sigma}, \tilde{T}] = 0$$

so that

$$(\tilde{T} \widetilde{P_\sigma})^2 = (\widetilde{C_\sigma} \tilde{T} \widetilde{P_{\sigma'}})^2 = \mathbf{1}.$$

Finally, let us observe that $\widetilde{P_\sigma}$ is a *canonical symmetry* in $\mathcal{H}$ [16], which immediately generates orthogonal canonical projectors

$$\tilde{\Pi}^{(\pm)} = \tfrac{1}{2}(\mathbf{1} \pm \widetilde{P_\sigma})$$

and a canonical decomposition

$$\mathcal{H} = \tilde{\Pi}^{(+)}\mathcal{H} \oplus \tilde{\Pi}^{(-)}\mathcal{H}$$

defining a Krein space.

Remark. It is easily verified that $\text{Tr}\widetilde{P_\sigma}$ depends only on the number of simple Jordan blocks $J_a(E_{n_0})$ of odd dimension associated with the real eigenvalues appearing in the Jordan canonical form of $\tilde{H}$. Hence, by choosing a suitable sequence (that we denote again with σ) so that alternate signs + and − are associated with such blocks, one easily sees that

$$\text{Tr}\widetilde{P_\sigma} = 0 \quad \text{if the space is } even \text{ dimensional}$$
$$\text{Tr}\widetilde{P_\sigma} = 1 \quad \text{if the space is } odd \text{ dimensional}.$$

Moreover this choice of σ maximizes the number of arbitrary parameters of the most general (real, symmetric) parity matrix in the sense of [21]. Henceforth we will usually refer to such a σ, and will just denote the generalized parity and charge-conjugation operators by P and C for the sake of simplicity.

5. Indefinite inner product spaces and symmetries of H

The arguments in the preceding section led us in a natural way to discuss the properties of indefinite inner product spaces. In this section we go on to the symmetries of these spaces, where the scalar product is defined as in equation (24); moreover, since equation (1) holds with any P_σ in place of η, considering the peculiar properties of the generalized parity operators (see, for instance, equations (21) and (22)), and remembering the remark at the end of the preceding section, we will always put P instead of η in (24). Furthermore, we will investigate the connections between these symmetries and the previously obtained involutory C, TP and CTP symmetries of H.

The set $\mathcal{S}$ of the symmetries in indefinite metric spaces (that is the transformations preserving the modulus of the indefinite scalar product) was investigated in [9], where a generalized Wigner's theorem was proved, and the following fourfold classification was obtained:

(1) An operator U such that

$$\langle\psi|U^\dagger PU|\phi\rangle = \langle\psi|P|\phi\rangle \tag{33}$$

is called a P-unitary operator in the indefinite metric space.
(2) An operator V such that

$$\langle\psi|V^\dagger PV|\phi\rangle = \langle\phi|P|\psi\rangle \tag{34}$$

is called a P-antiunitary operator in the indefinite metric space.
(3) An operator U such that

$$\langle\psi|U^\dagger PU|\phi\rangle = -\langle\psi|P|\phi\rangle \tag{35}$$

is called a P-pseudounitary operator in the indefinite metric space.
(4) An operator V such that

$$\langle\psi|V^\dagger PV|\phi\rangle = -\langle\phi|P|\psi\rangle \tag{36}$$

is called a P-pseudoantiunitary operator in the indefinite metric space.

Further, we recall that the spectral properties of P-unitary operators were already investigated in [8].

Let us then focus our attention on a relevant subset of $\mathcal{S}_H \subset \mathcal{S}$, that is the set of all the invertible P-unitary, P-antiunitary, P-pseudounitary and P-pseudoantiunitary operators $\mathcal{S}$ which commute with H:

$$\mathcal{S}_H = \{X \in \mathcal{S} : [H, X] = 0\}.$$

First of all, with regard to the P-unitary symmetries U of $\mathcal{S}_H$, we observe that all of them can be explicitly obtained by considering the linear operators X belonging to the commutant of H [22] and then imposing the constraints (33). By a simple calculation one easily recognizes that the operator C_σ given in equation (15) is a P-unitary operator.

Let us now come to the P-antiunitary, P-pseudounitary and P-pseudoantiunitary symmetry operators. We have already seen that an involutory metric $\widetilde{P}$ can always be obtained by

a suitable transformation S (see equations (27) and (25)), and this can be done without spoiling the assumptions of the generalized Wigner theorem [9]. Hence, all the (P-unitary,) P-antiunitary, P-pseudounitary and P-pseudoantiunitary symmetry operators (whenever they exist) are in a one-to-one correspondence with the ($\tilde{P}$-unitary,) $\tilde{P}$-antiunitary, $\tilde{P}$-pseudounitary and $\tilde{P}$-pseudoantiunitary symmetries, respectively.

Recalling the peculiar properties of the antilinear operators $\tilde{T}\tilde{P}$ and $\tilde{C}\tilde{T}\tilde{P}$ (namely, they are Hermitian, involutory operators which commute with $\tilde{P}$), and coming back to the basis $\mathfrak{E}$, in virtue of the above correspondence, the following proposition easily follows, which paraphrases a similar statement in [9]:

Proposition 3. *Any P-antiunitary symmetry operator $V \in S$ can be written as follows:*

$$V = (CTP)U = (TP)U'$$

where U and U' represent P-unitary symmetries of $\mathcal{H}$ and CTP and TP are the involutory operators given in equations (18) and (17), respectively.

In particular, taking into account equations (21) and (22), one can prove immediately the following corollary, which allows us to obtain all the P-antiunitary elements of $\mathcal{S}_H$ if the P-unitary ones are known.

Corollary. *Any P-antiunitary symmetry operator $V \in \mathcal{S}_H$ can be written as follows:*

$$V = (CTP)U = (TP)U'$$

where U and U' represent P-unitary symmetries of H and CTP and TP are the involutory operators given in equations (18) and (17), respectively.

As to $\widetilde{P}$-pseudounitary and $\widetilde{P}$-pseudoantiunitary operators, they may exist only in indefinite metric spaces in which the eigenvalues $+1$ and -1 of the metric operator $\widetilde{P}$ have the same multiplicity [9] (i.e., $\mathrm{Tr}\widetilde{P} = 0$).

Hence, from now on we limit ourselves to considering even-dimensional spaces and we choose in $\{P_\sigma\}$ a P which is congruent to a traceless, involutory operator $\tilde{P}$ (see the remark in section 4). Then, it is not difficult to see that any P-pseudounitary operator can be obtained by multiplying the P-unitary ones by an involutory operator R (also called *P-reflecting operator*) which satisfies the following condition:

$$R^\dagger P R = -P$$

and a similar statement holds for P-pseudoantiunitary operators.

In this connection, the following proposition provides a necessary and sufficient condition for a P-pseudo-Hermitian Hamiltonian H to admit P-pseudounitary and P-pseudoantiunitary symmetries.

Proposition 4. *Let H be a P-pseudo-Hermitian operator with a discrete spectrum, where P is congruent to a traceless, involutory operator. Then the following conditions are equivalent:*

 (i) *the Jordan blocks associated with any real eigenvalue of H occur in pairs;*
 (ii) *an involutory operator R exists such that $[H, R] = 0$ and $R^\dagger P R = -P$;*
(iii) *an antilinear operator $\mathfrak{T}$ exists such that $[H, \mathfrak{T}] = 0$ and $\mathfrak{T}^2 = -1$.*

Proof. Let us prove the implication (i) $\Rightarrow$ (ii).

Let condition (i) hold; then, we can write

$$
\begin{aligned}
H = \sum_{n_0} \sum_{a=1}^{\frac{d_{n_0}}{2}} & \Bigg[E_{n_0} \sum_{i=1}^{p_{n_0,a}} \left(\big|\psi_{n_0}, a, i\big\rangle\big\langle\phi_{n_0}, a, i\big| + \Big|\psi_{n_0}, a + \frac{d_{n_0}}{2}, i\Big\rangle\Big\langle\phi_{n_0}, a + \frac{d_{n_0}}{2}, i\Big| \right) \\
& + \sum_{i=1}^{p_{n_0,a}-1} \left(\big|\psi_{n_0}, a, i\big\rangle\big\langle\phi_{n_0}, a, i+1\big| + \Big|\psi_{n_0}, a + \frac{d_{n_0}}{2}, i\Big\rangle\Big\langle\phi_{n_0}, a + \frac{d_{n_0}}{2}, i+1\Big| \right) \Bigg] \\
& + \sum_{n_+,n_-} \sum_{a=1}^{d_{n_+}} \Bigg[\sum_{i=1}^{p_{n_+,a}} \left(E_{n_+} \big|\psi_{n_+}, a, i\big\rangle\big\langle\phi_{n_+}, a, i\big| + E_{n_-} \big|\psi_{n_-}, a, i\big\rangle\big\langle\phi_{n_-}, a, i\big| \right) \\
& + \sum_{i=1}^{p_{n_+,a}-1} \left(\big|\psi_{n_+}, a, i\big\rangle\big\langle\phi_{n_+}, a, i+1\big| + \big|\psi_{n_-}, a, i\big\rangle\big\langle\phi_{n_-}, a, i+1\big| \right) \Bigg].
\end{aligned}
$$

Choosing P in the following form:

$$
\begin{aligned}
P = \sum_{n_0} \sum_{a=1}^{\frac{d_{n_0}}{2}} \sum_{i=1}^{p_{n_0,a}} & \left(\big|\phi_{n_0}, a, p_{n_0,a} + 1 - i\big\rangle\big\langle\phi_{n_0}, a, i\big| - \Big|\phi_{n_0}, a + \frac{d_{n_0}}{2}, p_{n_0,a} + 1 - i\Big\rangle \right. \\
& \left. \times \Big\langle\phi_{n_0}, a + \frac{d_{n_0}}{2}, i\Big| \right) + \sum_{n_+,n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} \sigma_{n_+}^a \left(\big|\phi_{n_+}, a, p_{n_+,a} + 1 - i\big\rangle\big\langle\phi_{n_-}, a, i\big| \right. \\
& \left. + \big|\phi_{n_-}, a, p_{n_+,a} + 1 - i\big\rangle\big\langle\phi_{n_+}, a, i\big| \right).
\end{aligned}
$$

one easily obtains by inspection that the operator

$$
\begin{aligned}
R = \sum_{n_0} \sum_{a=1}^{\frac{d_{n_0}}{2}} \sum_{i=1}^{p_{n_0,a}} & \left(\big|\psi_{n_0}, a, i\big\rangle\Big\langle\phi_{n_0}, a + \frac{d_{n_0}}{2}, i\Big| + \Big|\psi_{n_0}, a + \frac{d_{n_0}}{2}, i\Big\rangle\big\langle\phi_{n_0}, a, i\big| \right) \\
& + \sum_{n_+,n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} \left(\big|\psi_{n_+}, a, i\big\rangle\big\langle\phi_{n_+}, a, i\big| - \big|\psi_{n_-}, a, i\big\rangle\big\langle\phi_{n_-}, a, i\big| \right)
\end{aligned} \tag{37}
$$

is involutory and satisfies the conditions $R^\dagger P R = -P$ and $[H, R] = 0$.

(ii) $\Rightarrow$ (i). Let us assume that condition (ii) holds; then moving from $\mathfrak{E}$ to the orthonormal basis $\mathfrak{F} = \{|u_n, a, i\rangle\}$, conditions (ii) become

$$
[\widetilde{H}, \widetilde{R}] = 0 \qquad \text{and} \qquad \widetilde{R}^\dagger \widetilde{P} \widetilde{R} = -\widetilde{P} \tag{38}
$$

where $\widetilde{H}$ and $\widetilde{P}$ are given in equations (28) and (30), respectively, and $\widetilde{R} = S^{-1} R S$.

With reference to the block-diagonal form of $\widetilde{H}$, we denote by $\mathcal{E}_n$ the vector space associated with the eigenvalue E_n, i.e., the space spanned by the set of vectors $\{|u_n, a, i\rangle, a = 1, \ldots, d_n, i = 1, \ldots, p_{n,a}\}$. Then, we observe that, since $[\widetilde{H}, \widetilde{R}] = 0$, the matrix elements of $\tilde{R}$ between states belonging to two different spaces $\mathcal{E}_n$ and $\mathcal{E}_{n'}$ are always zero [22], hence $\tilde{R}$ maps each $\mathcal{E}_n$ onto itself.

Now, let us consider a vector space $\mathcal{E}_{n_0}$ associated with a real eigenvalue E_{n_0}, and let us pick out in it the subspace spanned by the set $\{|u_{n_0}, a, i\rangle, i = 1, \ldots, p_{n_0,a}\}$, with a fixed a. One immediately realizes that $\widetilde{R}|u_{n_0}, a, 1\rangle$ is again an eigenvector of $\widetilde{H}$, corresponding to the same eigenvalue; furthermore, the vectors $\{\tilde{R}|u_{n_0}, a, i\rangle, i = 1, \ldots, p_{n_0,a}\}$ belong to $\mathcal{E}_{n_0}$ and they

are easily seen to be linearly independent. Applying $\tilde{R}$ on the left-hand side in equation (29), and using again $[\tilde{H}, \tilde{R}] = 0$, one obtains also

$$\tilde{H}\tilde{R}|u_n, a, i\rangle = E_n\tilde{R}|u_n, a, i\rangle + \tilde{R}|u_n, a, i-1\rangle \qquad i \neq 1.$$

We can conclude that the vectors $\{\tilde{R}|u_{n_0}, a, i\rangle, i = 1, \ldots, p_{n_0,a}\}$ actually span the subspace associated with $J_{a'}(E_{n_0})$, hence dim $J_a(E_{n_0}) = $ dim $J_{a'}(E_{n_0})$, and the two blocks are identical.

Then, if *for all values* of the degeneracy label a it happens that $a \neq a'$, condition (i) follows at once.

In contrast, let us suppose *ab absurdo* that an a exists such that $\tilde{R}$ maps the subspace spanned by $\{|u_{n_0}, a, i\rangle, i = 1, \ldots, p_{n_0,a}\}$ onto itself. In this case, since $\tilde{R}|u_{n_0}, a, 1\rangle$ is an eigenvector of $\tilde{H}$ and $\tilde{R}^2 = \mathbf{1}$, it follows that:

$$\tilde{R}|u_{n_0}, a, 1\rangle = \epsilon|u_{n_0}, a, 1\rangle \qquad \text{with} \quad \epsilon = \pm 1.$$

Then, by the same iterative procedure we used in the proof of proposition 1, we can prove that

$$\tilde{R}|u_{n_0}, a, i\rangle = \epsilon|u_{n_0}, a, i\rangle \qquad i = 1, \ldots, p_{n_0,a}. \tag{39}$$

Taking now into account the condition $\tilde{R}^\dagger\tilde{P}\tilde{R} = -\tilde{P}$, and going on to evaluate the corresponding matrix elements, from equation (39) and its adjoint we obtain

$$\langle u_{n_0}, a, j|\tilde{R}^\dagger\tilde{P}\tilde{R}|u_{n_0}, a, i\rangle = \langle u_{n_0}, a, j|\epsilon\tilde{P}\epsilon|u_{n_0}, a, i\rangle = \langle u_{n_0}, a, j|\tilde{P}|u_{n_0}, a, i\rangle$$

$$= -\langle u_{n_0}, a, j|\tilde{P}|u_{n_0}, a, i\rangle = 0 \qquad \text{for all } i, j = 1, \ldots, p_{n_0,a},$$

which leads to a contradiction, since equation (30) implies $\langle u_{n_0}, a, p_{n_0,a} + 1 - i|\tilde{P}|u_{n_0}, a, i\rangle = \sigma_{n_0}^a \neq 0$, for all $i = 1, \ldots, p_{n_0,a}$.

Hence the Jordan blocks associated with any real eigenvalue of $\tilde{H}$ must occur in pairs, which in turn implies that the same happens for the Jordan blocks associated with any real eigenvalue of H.

The equivalence (i) $\Leftrightarrow$ (iii) was proved in [11]. We only recall that the P-pseudoantiunitary operator $\mathfrak{T}$ which commutes with H and satisfies the condition $\mathfrak{T}^2 = -\mathbf{1}$ is given by

$$\mathfrak{T} = \sum_{n_0} \sum_{a=1}^{\frac{d_{n_0}}{2}} \sum_{i=1}^{p_{n_0,a}} \left(|\psi_{n_0}, a, i\rangle K \left\langle \phi_{n_0}, a + \frac{d_{n_0}}{2}, i\right| - \left|\psi_{n_0}, a + \frac{d_{n_0}}{2}, i\right\rangle K \langle\phi_{n_0}, a, i| \right)$$

$$+ \sum_{n_+, n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_0,a}} \left(|\psi_{n_+}, a, i\rangle K\langle\phi_{n_-}, a, i| - |\psi_{n_-}, a, i\rangle K\langle\phi_{n_+}, a, i| \right) \tag{40}$$

hence, it is P-pseudoantiunitary and coincides with the product RTP. □

6. A physical example: the Mashhoon–Papini Hamiltonian

We finally apply the general formalism developed in the preceding sections to the following pseudo-Hermitian Hamiltonian:

$$H_{\text{eff}} = \begin{pmatrix} E & ir \\ -is & E \end{pmatrix} \qquad (E, r, s \in \mathbb{R})$$

which arises in the Mashhoon–Papini model, where one introduces a (time-reversal violating) spin-rotation coupling to explain the muon's anomalous g factor [23].

Though it is elementary, this Hamiltonian, which has been extensively studied elsewhere by some of the authors [11, 24], is quite fit to illustrate the above results.

Indeed, the eigenvalues of H_{eff} are

$$E_{1,2} = E \pm \sqrt{rs}$$

and depending on the values of r, s three different cases can occur.

6.1. Case of real, non-degenerate spectrum

Whenever $rs \neq 0$, H_{eff} is diagonalizable and the biorthonormal eigenbasis is given by

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i\chi^{\frac{1}{2}} \\ 1 \end{pmatrix} \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i\chi^{\frac{1}{2}} \\ 1 \end{pmatrix}$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i\chi^{-\frac{1}{2}} \\ 1 \end{pmatrix} \qquad |\phi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i\chi^{-\frac{1}{2}} \\ 1 \end{pmatrix}$$

where we omit the useless labels a, i and put $\chi = \frac{r}{s}$. Moreover, if χ is positive, H_{eff} admits a non-degenerate real spectrum and a (traceless) generalized parity (see equation (14)) is given by

$$P = |\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2| = \chi^{-\frac{1}{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

whereas the charge-conjugation and the time-reversal operators are

$$C = |\psi_1\rangle\langle\phi_1| - |\psi_2\rangle\langle\phi_2| = i \begin{pmatrix} 0 & \chi^{\frac{1}{2}} \\ -\chi^{-\frac{1}{2}} & 0 \end{pmatrix}$$

$$T = |\psi_1\rangle K \langle\psi_1| + |\psi_2\rangle K \langle\psi_2| = \begin{pmatrix} -\chi & 0 \\ 0 & 1 \end{pmatrix} K.$$

Note that we have chosen the same sequence of signs both in P and C. Furthermore, we recall that now T is not physically meaningful since we are dealing with a fermionic Hamiltonian.

By a direct calculation the P-unitary symmetries U of H_{eff} can be obtained,

$$U = e^{i\alpha}|\psi_1\rangle\langle\phi_1| + e^{i\beta}|\psi_2\rangle\langle\phi_2| = \frac{1}{2} \begin{pmatrix} (e^{i\alpha} + e^{i\beta}) & i\chi^{\frac{1}{2}}(e^{i\alpha} - e^{i\beta}) \\ -i\chi^{-\frac{1}{2}}(e^{i\alpha} - e^{i\beta}) & (e^{i\alpha} + e^{i\beta}) \end{pmatrix} \tag{41}$$

$(\alpha, \beta \in \mathbb{R})$, whereas the generators of the P-antiunitary symmetries of H_{eff} are

$$TP = |\psi_1\rangle K \langle\phi_1| - |\psi_2\rangle K \langle\phi_2| = -i \begin{pmatrix} 0 & \chi^{\frac{1}{2}} \\ \chi^{-\frac{1}{2}} & 0 \end{pmatrix} K$$

$$CTP = |\psi_1\rangle K \langle\phi_1| + |\psi_2\rangle K \langle\phi_2| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} K.$$

We stress the fact that in this case no degeneracy of the real eigenvalues occurs, hence H_{eff} cannot admit P-pseudounitary symmetries (see proposition 4).

A positive definite metric can be easily computed (see equation (23))

$$P_+ = PC = \begin{pmatrix} \chi^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Observing that $[P, C] = 0$, P-unitarity and P_+-unitarity coincide. In particular, the (non-unitary) time evolution operator $U(t) = e^{-iHt}$ [24] is trivially P_+-unitary, and indeed it has the form (41) with $-E_1 t$ in place of α and $-E_2 t$ in place of β.

Remark. It is worthwhile stressing that in the case under consideration the eigenvalues of $U(t)$ are $e^{i\alpha}$ and $e^{i\beta}$ (see (41)), hence, they are unimodular. Now, it is well known that in general the eigenvalues of a P-unitary operator either are unimodular or they occur in pairs $\lambda, \frac{1}{\lambda^*}$ [8, 17]; nevertheless, whenever a positive definite inner product can be introduced, one could easily prove that only the first alternative can occur, by employing the same techniques as in the textbooks but merely using the new inner product (24) in place of the usual one (see also the discussion at the beginning of section 4).

In a similar way, one can define a transition probability between the states $|\omega\rangle$ and $|\omega'\rangle$ by putting, in analogy with the ordinary quantum mechanical prescription

$$\mathcal{P}_{\omega \to \omega'}(t) = \left| \langle\!\langle \omega, U(t)\omega' \rangle\!\rangle_{P_+} \right|^2.$$

For instance, assuming the initial condition $|\psi(0)\rangle = \binom{0}{1} \equiv |\psi_-\rangle$ and recalling that $|\psi_+\rangle \equiv \binom{1}{0}$ [23, 24], the spin-flip probability turns out to be

$$\mathcal{P}_{\psi_- \to \psi_+}(t) = \tfrac{1}{2}(1 - \cos 2\sqrt{rst})$$

which fully agrees (without any approximation) with that proposed in [23] in order to interpret the discrepancy between the experimental and the standard model values of the muon's anomalous g-factor. (We recall that some of the authors proposed elsewhere [24] a different form for the above probability, with an *ad hoc* normalization factor in order to consider the non-unitarity of $U(t)$).

6.2. Case of complex, non-degenerate spectrum

If χ is negative, H_{eff} admits two complex-conjugate eigenvalues. In this case P and C are uniquely defined (up to a global sign), and we have

$$P = |\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1| = \begin{pmatrix} \chi^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = |\psi_1\rangle\langle\phi_1| + |\psi_2\rangle\langle\phi_2| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T = |\psi_1\rangle K \langle\psi_1| + |\psi_2\rangle K \langle\psi_2| = \begin{pmatrix} -\chi & 0 \\ 0 & 1 \end{pmatrix} K.$$

Note that the charge-conjugation operator C becomes the identity (see equation (15); anyway, an involutory, non-trivial symmetry exists, in the sense of proposition 1 and of equation (12), and it coincides with the operator C in section 6.1).

The P-unitary symmetries of H_{eff} are given by

$$U = u|\psi_1\rangle\langle\phi_1| + \frac{1}{u^*}|\psi_2\rangle\langle\phi_2| = \frac{1}{2u^*} \begin{pmatrix} (|u|^2 + 1) & i\chi^{\frac{1}{2}}(|u|^2 - 1) \\ -i\chi^{-\frac{1}{2}}(|u|^2 - 1) & (|u|^2 + 1) \end{pmatrix}$$

where $u \in \mathbb{C}$ (see also the remark in section 6.1) and the generator of the P-antiunitary symmetries of H_{eff} is given by

$$TP = CTP = |\psi_1\rangle K \langle\phi_1| + |\psi_2\rangle K \langle\phi_2| = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} K.$$

Moreover, in this case H_{eff} also admits P-pseudounitary and P-pseudoantiunitary symmetries. The P-reflecting operator assumes the form (see equation (37))

$$R = |\psi_1\rangle\langle\phi_1| - |\psi_2\rangle\langle\phi_2| = \begin{pmatrix} 0 & -|\chi|^{\frac{1}{2}} \\ -|\chi|^{-\frac{1}{2}} & 0 \end{pmatrix}$$

(by a simple calculation it is easy to verify that $R^\dagger PR = -P$); finally, the P-pseudoantiunitary operator $\mathfrak{T}$ which commutes with H_{eff} and satisfies the condition $\mathfrak{T}^2 = -\mathbf{1}$ is

$$\mathfrak{T} = |\psi_1\rangle K \langle\phi_2| - |\psi_2\rangle K \langle\phi_1| = \begin{pmatrix} 0 & -|\chi|^{\frac{1}{2}} \\ |\chi|^{-\frac{1}{2}} & 0 \end{pmatrix} K.$$

6.3. Case of real, degenerate spectrum

If an off-diagonal term in H_{eff} is zero, the Hamiltonian is no longer diagonalizable. Let us consider for instance $s = 0$:

$$H_{\text{eff}} = \begin{pmatrix} E & \mathrm{i}r \\ 0 & E \end{pmatrix} \qquad (E, r \in \mathbb{R}).$$

Omitting now the useless labels n, a the biorthonormal basis associated with H_{eff} is

$$|\psi_{,1}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\psi_{,2}\rangle = \frac{\mathrm{i}}{r}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad |\phi_{,1}\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |\phi_{,2}\rangle = \mathrm{i}r\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and the generalized parity, charge-conjugation and time-reversal operators, respectively, are

$$P = |\phi_{,1}\rangle\langle\phi_{,2}| + |\phi_{,2}\rangle\langle\phi_{,1}| = \mathrm{i}r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$C = |\psi_{,1}\rangle\langle\phi_{,1}| + |\psi_{,2}\rangle\langle\phi_{,2}| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$T = |\psi_{,1}\rangle K \langle\psi_{,2}| + |\psi_{,2}\rangle K \langle\psi_{,1}| = \frac{\mathrm{i}}{r}\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} K.$$

(Note that $C \equiv \mathbf{1}$, see proposition 1.)

The generator of the P-antiunitary symmetries of H_{eff} is given by

$$TP = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} K.$$

while the form of the P-unitary symmetries of H_{eff} is

$$U = \mathrm{e}^{\mathrm{i}\alpha}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \qquad \alpha, p \in R.$$

7. Summary and conclusions

In the first part of this paper, we extended to the nondiagonalizable case the definitions of generalized parity, charge-conjugation and time-reversal operators which can be associated with any pseudo-Hermitian Hamiltonian H, and wrote them explicitly in terms of the elements of the biorthonormal eigenbasis of H.

The generalized operators given above, and in particular TP and CTP, play a fundamental role in a seemingly very different context. Indeed, we considered in section 4 *all* the symmetries of the Krein space that one can associate with a P-pseudo-Hermitian Hamiltonian H. They include P-antiunitary and (possibly) P-pseudounitary and P-pseudoantiunitary symmetries, besides the P-unitary ones which have been already introduced [7, 8]. Then, in this connection, we have proved that any P-pseudo-Hermitian Hamiltonian H admits P-antiunitary symmetries and the generators TP and CTP of such symmetries are explicitly shown in equations (17) and (18), respectively. Moreover, TP and CTP also generate all the P-antiunitary symmetries of the space $\mathcal{H}$.

Furthermore, if a P-reflecting operator R which commutes with H exists in the Krein space, RTP generates all the P-pseudoantiunitary symmetries of the Hamiltonian. In particular, proposition 4 provides a necessary and sufficient condition for the existence of P-pseudounitary and P-pseudoantiunitary symmetries of H, and their generators R and RTP

are given in equations (37) and (40), respectively. We stress here that RTP coincides with the antilinear operator $\mathfrak{T}$ which has the same properties as the time-reversal operator of fermionic systems (see the discussion at the end of section 3), so that proposition 4 links in a perhaps unexpected way the existence of a time-reversal symmetry of H and of a P-reflecting symmetry of the Krein space. We remark, however, that proposition 4 does not apply whenever a positive definite operator (e.g., P_+) is chosen to define the new inner product (24).

Finally, some hints arose when these concepts were applied to the study of a physical system (the Mashhoon–Papini Hamiltonian), in particular regarding the non-unitary evolution of such a system.

We recall that Krein spaces have already been introduced in connection with pseudo-Hermitian Hamiltonians, in the attempt to enlarge the framework of quantum mechanics, and several interpretations of the vectors in $\mathcal{H}^{(-)}$ have been proposed [1, 25]. We believe that this paper fits very well in this attempt, in that it allows one to extend the above description to the nondiagonalizable case, and to better understand the symmetry properties of such spaces.

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